Supplementary Technical Document

1 derivation of heat transfer equation

Following [1] we derive the heat equation from the first and second laws of thermodynamics. The Lagrangian form of the laws are given below:

- (first law of thermodynamics) energy balance

$$\dot{\Phi} = \mathbf{P} : \dot{\mathbf{F}} - \nabla \cdot \mathbf{Q} + R_0(\mathbf{X}) r(\mathbf{X}, t) \tag{1}$$

- (second law of thermodynamics) Clausius-Duhem inequality

$$\Theta \dot{\eta}_m \geq R_0 r - \nabla \cdot \mathbf{Q} + \Theta^{-1}(\nabla \Theta \cdot \mathbf{Q}) \tag{2}$$

Where the dot symbol stands for the material derivative, Φ is the internal energy density per unit volume, $\mathbf{P}$ is the first Piola-Kirchhoff stress, $\mathbf{F}$ is the deformation gradient, $\mathbf{Q}$ is the material heat flux field defined through the Fourier-Stokes heat flux $\mathbf{q}(\mathbf{x}, t)$ by $\mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q}$ with $J = \det(\mathbf{F})$, $R_0(\mathbf{X}) = R(\mathbf{X}, 0) = R(\mathbf{X}, t)J(\mathbf{X}, t)$ is material density at rest configuration, r is body heating supply, Θ is the absolute temperature, and η_m is the entropy density per unit volume. The free energy density is defined as

$$\Psi(\mathbf{X}, t) = \Phi(\mathbf{X}, t) - \Theta(\mathbf{X}, t)\eta_m(\mathbf{X}, t).$$

For thermoelastic solids we have the following assumptions:

- The functions Φ, Ψ and η_m can be represented as functions of $\mathbf{F}$ and Θ. Furthermore we have

$$\frac{\partial \Phi}{\partial \Theta} > 0, \qquad \frac{\partial \eta_m}{\partial \Theta} > 0$$

for any $\mathbf{F}$ with $\det(\mathbf{F}) > 0$ and $\Theta > 0$.

- The material heat flux vector $\mathbf{Q}$ is related to $\mathbf{F}$ and Θ by

$$\mathbf{Q} = -\mathbf{K}(\mathbf{F}, \Theta)\nabla\Theta$$

where $\mathbf{K}$ is called the thermal conductivity function, and the gradient is with respect to $\mathbf{X}$.

These assumptions combined with the two laws yield:

$$\begin{aligned}
\dot{\Psi} &= \dot{\Phi} - \dot{\Theta}\eta_m - \Theta\dot{\eta}_m \\
&= \mathbf{P} : \dot{\mathbf{F}} + (R_0 r - \nabla \cdot \mathbf{Q}) - \dot{\Theta}\eta_m - \Theta\dot{\eta}_m \\
&\leq \mathbf{P} : \dot{\mathbf{F}} + (\Theta\dot{\eta}_m - \Theta^{-1}(\nabla\Theta \cdot \mathbf{Q})) - \dot{\Theta}\eta_m - \Theta\dot{\eta}_m \\
&= \mathbf{P} : \dot{\mathbf{F}} + \Theta^{-1}(\nabla\Theta \cdot \mathbf{K}\nabla\Theta) - \dot{\Theta}\eta_m.
\end{aligned}$$

Since $\Psi = \Psi(\mathbf{F}, \Theta)$ we can write

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial\Psi}{\partial\Theta}\dot{\Theta}.$$

Then the above equation becomes

$$\left(\frac{\partial\Psi}{\partial\mathbf{F}} - \mathbf{P}\right) : \dot{\mathbf{F}} + \left(\eta_m + \frac{\partial\Psi}{\partial\Theta}\right)\dot{\Theta} - \Theta^{-1}(\nabla\Theta \cdot \mathbf{K}\nabla\Theta) \leq 0. \tag{3}$$

Since $\mathbf{F}$, Θ and their derivatives are independent of each other, we must have $\mathbf{P} = \frac{\partial\Psi}{\partial\mathbf{F}}$, $\eta_m = -\frac{\partial\Psi}{\partial\Theta}$, and $\mathbf{K}$ is positive semi-definite. Note that the first equation coincides with the definition of the first Piola-Kirchhoff stress. We take $\mathbf{q} = \kappa\nabla\theta$, with a material heat conductivity constant $\kappa > 0$. With this formulation of $\mathbf{q}$ we have

$$\mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q} = \kappa J\mathbf{F}^{-1}\nabla_{\mathbf{x}}\theta = \kappa J\mathbf{F}^{-1}\mathbf{F}^{-T}\nabla_{\mathbf{X}}\Theta.$$

So $\mathbf{K} = \kappa J\mathbf{F}^{-1}\mathbf{F}^{-T}$ satisfies the positive semi-definite constraint.

We define the specific heat at constant volume α as

$$\begin{aligned}
0 < \alpha &= \frac{1}{R_0(X)}\frac{\partial\Phi}{\partial\Theta} = \frac{1}{R_0}\left(\frac{\partial\Psi}{\partial\Theta} + \eta_m + \Theta\frac{\partial\eta_m}{\partial\Theta}\right) \\
&= \frac{1}{R_0}\Theta\frac{\partial\eta_m}{\partial\Theta} = -\frac{1}{R_0}\Theta\frac{\partial^2\Psi}{\partial\Theta^2}.
\end{aligned}$$

Note that then we need $\frac{\partial^2\Psi}{\partial\Theta^2} < 0$.

In multiplicative plasticity theory the deformation gradient $\mathbf{F}$ is separated into the elastic component $\mathbf{F}^E$ and the plastic (or viscoplastic) part $\mathbf{F}^P$, so we have $\dot{\mathbf{F}} = \dot{\mathbf{F}}^E\mathbf{F}^P + \mathbf{F}^E\dot{\mathbf{F}}^P$, which yields

$$\dot{\mathbf{F}}^E = (\dot{\mathbf{F}} - \mathbf{F}^E\dot{\mathbf{F}}^P)(\mathbf{F}^P)^{-1}$$

Now we view Ψ as a function of $\mathbf{F}^E$ and Θ. Following the same steps as before we can derive a similar inequality as Equation (3):

$$\left(\frac{\partial\Psi}{\partial\mathbf{F}^E}(\mathbf{F}^P)^{-T} - \mathbf{P}\right) : \dot{\mathbf{F}} + \left(\eta_m + \frac{\partial\Psi}{\partial\Theta}\right)\dot{\Theta}$$
$$-\Theta^{-1}(\nabla\Theta\cdot\mathbf{K}\nabla\Theta) - \frac{\partial\Psi}{\partial\mathbf{F}^E}(\mathbf{F}^P)^{-T}(\mathbf{F}^E)^T : \dot{\mathbf{F}}^P \leq 0.$$

Again we get $\mathbf{P} = \frac{\partial\Psi}{\partial\mathbf{F}^E}(\mathbf{F}^P)^{-T}$ which matches the definition of the first Piola-Kirchhoff stress, $\eta_m = -\frac{\partial\Psi}{\partial\Theta}$, and $\mathbf{K}$ is positive semi-definite. The last term stands for plastic energy dissipation, and we need:

$$\mathbf{P}(\mathbf{F}^E)^T : \dot{\mathbf{F}}^P \geq 0.$$

Going back to the energy balance equation, we can rewrite it as

$$\begin{aligned}
0 &= \dot{\Phi} - (\dot{\Psi} + \dot{\Theta}\eta_m + \Theta\dot{\eta_m}) \\
&= (\mathbf{P} : \dot{\mathbf{F}} - \nabla\cdot\mathbf{Q} + R_0 r) - \left(\frac{\partial\Psi}{\partial\mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial\Psi}{\partial\Theta}\dot{\Theta} + \dot{\Theta}\eta_m + \Theta\dot{\eta}_m\right) \\
&= (\mathbf{P} : \dot{\mathbf{F}} - \nabla\cdot\mathbf{Q} + R_0 r) - (\mathbf{P} : \dot{\mathbf{F}} - \eta_m\dot{\Theta} + \dot{\Theta}\eta_m + \Theta\dot{\eta}_m) \\
&= -\nabla\cdot\mathbf{Q} + R_0 r - \Theta\dot{\eta}_m \\
&= -\nabla\cdot\mathbf{Q} + R_0 r - \Theta\left(\frac{\partial\eta_m}{\partial\mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial\eta_m}{\partial\Theta}\dot{\Theta}\right) \\
&= \nabla\cdot(\mathbf{K}\nabla\Theta) + R_0 r + \Theta\frac{\partial^2\Psi}{\partial\mathbf{F}\partial\Theta} : \dot{\mathbf{F}} - R_0\alpha\dot{\Theta}.
\end{aligned}$$

To further simplify the equation we assume there is no body heating supply, i.e. $r = 0$, leading to the heat equation

$$R_0\alpha\dot{\Theta} = \nabla\cdot(\mathbf{K}\nabla\Theta) + \Theta\frac{\partial^2\Psi}{\partial\mathbf{F}\partial\Theta} : \dot{\mathbf{F}}. \tag{4}$$

With our choice of Ψ we have

$$\frac{\partial^2\Psi}{\partial\Theta\partial\mathbf{F}} = -c\lambda\mathbf{F}^{-T}.$$

2 heat transfer discrete update

For any test function $U(\mathbf{X}, 0) : \Omega^0 \to \mathbb{R}$, we have

$$\int_{\Omega_0} R_0 \alpha \dot{\Theta} U d\mathbf{X} = \int_{\Omega_0} \nabla \cdot (\mathbf{K} \nabla \Theta) U + \Theta \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} : \dot{\mathbf{F}} U d\mathbf{X}. \tag{5}$$

Discretizing the left hand side with respect to time and space in the usual MPM style, and interpolating $U = \sum_j U_j(t) N_j(\mathbf{X})$, $\Theta = \sum_i \Theta_i(t) N_i(\mathbf{X})$ using shape functions $N_j(\mathbf{X})$, we get

$$\begin{aligned}
\int_{\Omega_0} R_0 \alpha \dot{\Theta} U d\mathbf{X} &= \int_{\Omega_0} R_0 \alpha \sum_i \dot{\Theta}_i N_i \sum_j U_j N_j d\mathbf{X} \\
&\approx \int_{\Omega_0} R(\mathbf{X}, t^n) J \alpha \sum_i \frac{\tilde{\Theta}_i^{n+1} - \Theta_i^n}{\Delta t} N_i \sum_j U_j N_j d\mathbf{X} \\
&\approx \int_{\Omega_t} \rho(\mathbf{x}, t^n) \alpha \sum_i \frac{\tilde{\theta}_i^{n+1} - \theta_i^n}{\Delta t} N_i \sum_j u_j N_j d\mathbf{x} \\
&= \frac{1}{\Delta t} \sum_{i,j} (\tilde{\theta}_i^{n+1} - \theta_i^n) u_j \int_{\Omega_t} \rho \alpha N_i N_j d\mathbf{x} \\
&\approx \frac{1}{\Delta t} \sum_j \left[\alpha m_j^n \tilde{\theta}_j^{n+1} - \alpha m_j^n \theta_j^n \right] u_j.
\end{aligned}$$

Here $\tilde{\theta}^{n+1}$ stands for the time t^{n+1} temperature but pulled back to the time t^n spatial configuration. In the last step we used the mass lumping strategy. For the right hand side using

$$\frac{\partial \mathbf{F}}{\partial t}(\mathbf{X}, t) = \frac{\partial}{\partial t} \frac{\partial \boldsymbol{\phi}(\mathbf{X}, t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{V}(\mathbf{X}, t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \nabla \mathbf{v} \mathbf{F},$$

we have

$$\begin{aligned}
\Theta \frac{\partial^2 \Psi}{\partial \mathbf{F} \partial \Theta} : \dot{\mathbf{F}} &= -\Theta c \lambda \mathbf{F}^{-T} : \nabla \mathbf{v} \mathbf{F} \\
&= -\Theta c \lambda \mathbf{I} : \nabla \mathbf{v} \\
&= -\Theta c \lambda \nabla \cdot \mathbf{v}.
\end{aligned}$$

The right hand side can then be rewritten as

$$
\begin{aligned}
&\int_{\Omega_0} U\nabla\cdot(\mathbf{K}\nabla\Theta) + \Theta\frac{\partial^2\Psi}{\partial\mathbf{F}\partial\Theta} : \dot{\mathbf{F}}U d\mathbf{X} \\
&= \int_{\partial\Omega_0} \mathbf{K}\nabla\Theta U\cdot\mathbf{N} ds(\mathbf{X}) - \int_{\Omega_0} (\mathbf{K}\nabla\Theta\cdot\nabla U + c\lambda\Theta\nabla\cdot\mathbf{v}U) d\mathbf{X} \\
&= -\int_{\partial\Omega_0} U\mathbf{Q}\cdot\mathbf{N} ds(\mathbf{X}) - \int_{\Omega_0} (\mathbf{K}\nabla\Theta\cdot\nabla U + c\lambda\Theta\nabla\cdot\mathbf{v}U) d\mathbf{X} \\
&= -\int_{\partial\Omega_t} u\mathbf{q}\cdot\mathbf{n} ds(\mathbf{x}) - \int_{\Omega_t} \left(\frac{1}{J}\mathbf{K}\mathbf{F}^T\nabla\theta\cdot\mathbf{F}^T\nabla u + \frac{1}{J}c\lambda\theta\nabla\cdot\mathbf{v}u\right) d\mathbf{x} \\
&= -\int_{\partial\Omega_t} u\mathbf{q}\cdot\mathbf{n} ds(\mathbf{x}) + \int_{\Omega_t} \left(\left(-\frac{1}{J}\mathbf{F}\mathbf{K}\mathbf{F}^T\nabla\theta\right)\cdot\nabla u - \frac{1}{J}c\lambda\theta\nabla\cdot\mathbf{v}u\right) d\mathbf{x} \\
&= -\int_{\partial\Omega_t} u\mathbf{q}\cdot\mathbf{n} ds(\mathbf{x}) + \int_{\Omega_t} \left(\mathbf{q}\cdot\nabla u - \frac{1}{J}c\lambda\theta\nabla\cdot\mathbf{v}u\right) d\mathbf{x}.
\end{aligned}
$$

The heat flow boundary condition is given in the world space as

$$\mathbf{q}\cdot\mathbf{n} = \beta(\theta - \theta_{out})$$

with some constant $\beta \geq 0$, where $\mathbf{n}$ is the outward normal, θ is the temperature of the object, and θ_{out} is the temperature of the outside. Note that when $\beta = 0$ this Robin boundary condition is reduced to Neumann, and when $\beta = \infty$ it becomes Dirichlet.

Discretizing with respect to space using shape functions we get

$$
\begin{aligned}
&-\int_{\partial\Omega_t} u\mathbf{q}\cdot\mathbf{n} ds(\mathbf{x}) + \int_{\Omega_t} \mathbf{q}\cdot\nabla u d\mathbf{x} - \int_{\Omega_t} \frac{1}{J}c\lambda\theta\nabla\cdot\mathbf{v}u d\mathbf{x} \\
&= -\int_{\partial\Omega_t} \sum_j u_j(t)N_j(\mathbf{x})\beta(\theta-\theta_{out}) ds(\mathbf{x}) + \int_{\Omega_t} \mathbf{q}\cdot\sum_j u_j\nabla N_j d\mathbf{x} - \int_{\Omega_t} \frac{1}{J}c\lambda\theta\nabla\cdot\mathbf{v}\sum_j u_j N_j d\mathbf{x} \\
&\approx -\sum_j u_j \sum_{p\in\partial\Omega_t} S_p\beta(\theta_p-\theta_{out})w_{jp} + \sum_j u_j \sum_p V_p\mathbf{q}_p\cdot\nabla w_{jp} - \sum_j u_j \sum_p \frac{1}{J_p}V_p c\lambda\theta_p\nabla\cdot\mathbf{v}_p w_{jp} \\
&= -\sum_j u_j \sum_{p\in\partial\Omega_t} S_p\beta(\theta_p-\theta_{out})w_{jp} + \sum_j u_j \sum_p V_{0p}(J_p\mathbf{q}_p\cdot\nabla w_{jp} - c\lambda\theta_p\nabla\cdot\mathbf{v}_p w_{jp})
\end{aligned}
$$

where the last equality is using the relation $V_p = J_p V_{0p}$ at each particle p, with V_{0p} being the rest state volume, $w_{jp} = N_j(\mathbf{x}_p)$, and S_p being the surface area of the local region tracked with partical p. Therefore the discrete form of the heat equation is given as

$$\frac{1}{\Delta t}\sum_j\left[\alpha m_j^n\tilde{\theta}_j^{n+1}-\alpha m_j^n\theta_j^n\right]u_j$$
$$=-\sum_j u_j\sum_{p\in\partial\Omega_t}S_p\beta(\theta_p-\theta_{out})w_{jp}+\sum_j u_j\sum_p V_{0p}(J_p\mathbf{q}_p\cdot\nabla w_{jp}-c\lambda\theta_p\nabla\cdot\mathbf{v}_p w_{jp}).$$

Taking $u_j=\delta_{ij}$, we get

$$\frac{1}{\Delta t}\left[\alpha m_i^n\tilde{\theta}_i^{n+1}-\alpha m_i^n\theta_i^n\right]=\sum_{p\in\partial\Omega_t}S_p\beta(\theta_p-\theta_{out})w_{ip}+\sum_p V_{0p}(J_p\mathbf{q}_p\cdot\nabla w_{ip}-c\lambda\theta_p\nabla\cdot\mathbf{v}_p w_{ip}).$$

This equation is used to perform the explicit update on $\alpha m\theta$ value at each time step. Note that if ignoring the kinetic term

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t}\rho\alpha\theta d\mathbf{x}&=\frac{d}{dt}\int_{\Omega_0}R(\mathbf{X},t)\alpha\Theta J d\mathbf{X}\\&=\frac{d}{dt}\int_{\Omega_0}R(\mathbf{X},0)\alpha\Theta d\mathbf{X}\\&=\int_{\Omega_0}R_0\alpha\dot{\Theta}d\mathbf{X}\\&=\int_{\Omega_0}\nabla\cdot(\mathbf{K}\nabla\Theta)d\mathbf{X}\\&=-\int_{\partial\Omega_0}\mathbf{Q}\cdot\mathbf{N}ds(\mathbf{X})\\&=-\int_{\partial\Omega_t}\mathbf{q}\cdot\mathbf{n}ds(\mathbf{x})\end{aligned}$$

So the left hand side can be regarded as the change of a total heat energy in the system. In order to conserve this energy we store θ on the particles and splat $\alpha m\theta$ from particles to grid nodes, along with the usual mass and linear momentum splat.

3 viscoplasticity return mapping

In this section we use $\mathbf{1}$ to denote the second order identity tensor (identity matrix), and $\mathbf{I}$ the fourth order identity tensor. For simplicity we denote our yield surface as

$$f(\boldsymbol{\tau})=\frac{p^2}{p_0^2}+q^2-M^2=0$$

where p_0 is a known value given porosity, and M stands for the yield stress. The yield surface is an elliptical shape centered at the origin, with radius measures $p_0 M$ and M. The position of each point on the curve can then be represented under polar coordinates by its angle $\alpha \in [0, \pi]$ from the positive p-axis, and p_0. In this sense for any point $(\alpha, \bar{p})$ in the plane, we can think of $\frac{\bar{p}-p_0}{p_0}$ as the "distance" of the point to the yield surface. More formally, we can define a canonical yield function $\tilde{f}$ as:

$$\tilde{f}(\alpha, \bar{p}) = \frac{\bar{p}}{p_0} - 1$$

Intuitively this is a signed distance function of the level set given by the yield surface. We use this notion as the yield criterion to simplify the computations in return mapping.

The return mapping is given by

$$\begin{aligned}
\boldsymbol{\tau}^{s,En+1} &= \text{argmin}_{\boldsymbol{\tau}} \frac{1}{2}(\boldsymbol{\tau}^{s,Etr} - \boldsymbol{\tau})^T \mathbf{C}^{-1}(\boldsymbol{\tau}^{s,Etr} - \boldsymbol{\tau}) + \frac{\Delta t}{\eta} g(f(\boldsymbol{\tau})) \\
&= \text{argmin}_{\boldsymbol{\tau}(\alpha,\bar{p})} \frac{1}{2}(\boldsymbol{\tau}^{s,Etr} - \boldsymbol{\tau})^T \mathbf{C}^{-1}(\boldsymbol{\tau}^{s,Etr} - \boldsymbol{\tau}) + \frac{\Delta t}{2\eta}(\frac{\bar{p}}{p_0} - 1)^2
\end{aligned}$$

We can write $\boldsymbol{\tau}, p, q$ in terms of α and $\bar{p}$:

$$\begin{aligned}
p &= \bar{p} M \cos(\alpha) \\
q &= M \sin(\alpha) \\
\boldsymbol{\tau} &= -p\mathbf{1} + \sqrt{\frac{2}{3}} q \frac{\boldsymbol{\epsilon}^{dev}}{||\boldsymbol{\epsilon}^{dev}||}
\end{aligned}$$

where $\boldsymbol{\epsilon}^{dev} = \boldsymbol{\epsilon} - \frac{1}{3} tr(\boldsymbol{\epsilon})\mathbf{1}$ is the deviatoric part of $\boldsymbol{\epsilon} = \log(\boldsymbol{\Sigma})$. $\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\epsilon}$ where $\mathbf{C} = 2\mu\mathbf{I} + \lambda\mathbf{1}\otimes\mathbf{1}$. We write the function to be minimized as $E(\alpha, \bar{p}) + \frac{\Delta t}{2\eta}(\frac{p}{p_0} - 1)^2$, and let $\mathbf{D} = \mathbf{C}^{-1}$, differentiating the above equation with respect to α and $\bar{p}$ we get

$$
\begin{aligned}
0 &= \frac{\partial E}{\partial \alpha} = \frac{\partial E}{\partial \boldsymbol{\tau}} : (\frac{\partial \boldsymbol{\tau}}{\partial p}\frac{\partial p}{\partial \alpha} + \frac{\partial \boldsymbol{\tau}}{\partial q}\frac{\partial q}{\partial \alpha}) \\
&= \mathbf{D}(\boldsymbol{\tau}^{s,Etr} - \boldsymbol{\tau}) : (\bar{p}M\sin(\alpha)\mathbf{1} + \sqrt{\frac{2}{3}}\frac{\boldsymbol{\epsilon}^{dev}}{||\boldsymbol{\epsilon}^{dev}||}M\cos(\alpha)) \\
&= (\boldsymbol{\epsilon}^{s,Etr} - \mathbf{D}\boldsymbol{\tau}) : (\bar{p}M\sin(\alpha)\mathbf{1} + \sqrt{\frac{2}{3}}\frac{\boldsymbol{\epsilon}^{dev}}{||\boldsymbol{\epsilon}^{dev}||}M\cos(\alpha)) \\
0 &= \frac{\partial E}{\partial \bar{p}} + \frac{\Delta t}{\eta}(\frac{\bar{p}}{p_0} - 1)\frac{1}{p_0} = \frac{\partial E}{\partial \boldsymbol{\tau}} : (\frac{\partial \boldsymbol{\tau}}{\partial p}\frac{\partial p}{\partial \bar{p}} + \frac{\partial \boldsymbol{\tau}}{\partial q}\frac{\partial q}{\partial \bar{p}}) + \frac{\Delta t}{\eta}(\frac{\bar{p}}{p_0} - 1)\frac{1}{p_0} \\
&= (\boldsymbol{\epsilon}^{s,Etr} - \mathbf{D}\boldsymbol{\tau}) : (-M\cos(\alpha)\mathbf{1}) + \frac{\Delta t}{\eta p_0}(\frac{\bar{p}}{p_0} - 1)
\end{aligned}
$$

Straight forward computation yields $\mathbf{D1} = \frac{1}{2\mu+3\lambda}\mathbf{1}$ and $\mathbf{D}\boldsymbol{\epsilon}^{dev} = \frac{1}{2\mu}\boldsymbol{\epsilon}^{dev}$. Note that $\mathbf{D} = \mathbf{C}^{-1}$ is symmetric, we move it to the other side of the contraction, The equations above then becomes

$$
\begin{aligned}
0 &= \boldsymbol{\epsilon}^{s,Etr} : (\bar{p}M\sin(\alpha)\mathbf{1} + \sqrt{\frac{2}{3}}\frac{\boldsymbol{\epsilon}^{dev}}{||\boldsymbol{\epsilon}^{dev}||}M\cos(\alpha)) \\
&\quad - \boldsymbol{\tau} : (\bar{p}M\sin(\alpha)\frac{1}{2\mu+3\lambda}\mathbf{1} + M\cos(\alpha)\sqrt{\frac{2}{3}}\frac{\boldsymbol{\epsilon}^{dev}}{2\mu||\boldsymbol{\epsilon}^{dev}||}) \\
&\quad = \bar{p}M\sin(\alpha)tr(\boldsymbol{\epsilon}^{s,Etr}) + \sqrt{\frac{2}{3}}\frac{M\cos(\alpha)\boldsymbol{\epsilon} : \boldsymbol{\epsilon}^{dev}}{||\boldsymbol{\epsilon}^{dev}||} \\
&\quad - \frac{\bar{p}M\sin(\alpha)}{2\mu+3\lambda}tr(\boldsymbol{\tau}) - \sqrt{\frac{2}{3}}\frac{M\cos(\alpha)\boldsymbol{\tau} : \boldsymbol{\epsilon}^{dev}}{2\mu||\boldsymbol{\epsilon}^{dev}||} \\
0 &= \boldsymbol{\epsilon}^{s,Etr} : (-M\cos(\alpha)\mathbf{1}) + \boldsymbol{\tau} : \frac{M\cos(\alpha)}{2\mu+3\lambda}\mathbf{1} + \frac{\Delta t}{\eta p_0}(\frac{\bar{p}}{p_0} - 1) \\
&\quad = -M\cos(\alpha)tr(\boldsymbol{\epsilon}) + \frac{M\cos(\alpha)}{2\mu+3\lambda}tr(\boldsymbol{\tau}) + \frac{\Delta t}{\eta p_0}(\frac{\bar{p}}{p_0} - 1)
\end{aligned}
$$

We further have $tr(\boldsymbol{\tau}) = -3p = -3\bar{p}M\cos(\alpha)$ and $\boldsymbol{\tau} : \boldsymbol{\epsilon}^{dev} = \sqrt{\frac{2}{3}}q||\boldsymbol{\epsilon}^{dev}|| = \sqrt{\frac{2}{3}}M\sin(\alpha)||\boldsymbol{\epsilon}^{dev}||$. Substitute the $\boldsymbol{\tau}$ terms with these expressions and grouping terms with respect to α and $\bar{p}$ we arrive at

$$
\begin{aligned}
0 &= A\bar{p}\sin(\alpha) + B\cos(\alpha) + C\bar{p}^2\sin(\alpha)\cos(\alpha) + D\sin(\alpha)\cos(\alpha) \\
0 &= F\cos(\alpha) + G\bar{p}\cos^2(\alpha) + H\bar{p}
\end{aligned}
$$

Where A, B, C, D, F, G, H are constant coefficients. Note that we can directly compute $\bar{p}$ from the second equation, plugging it back into the first equation we get $L(\alpha) = 0$ for some function L. We solve for α from this equation using Newton's method, and the corresponding $\bar{p}$ can be directly obtained. The projected stress $\tau^{s,En+1}$ is then computed with the new α and $\bar{p}$.

References

[1] O. Gonzalez and A. Stuart. *A first course in continuum mechanics.* Cambridge University Press, 2008.