Supplementary notes on return mapping and theory of plasticity

1 Return mapping

A trial state of deformation $\tilde{\mathbf{F}}^E$ is computed, assuming no plastic flow from time t^n to t^{n+1}. With this assumption, the plastic deformation does not change over the time step, so $\mathbf{F}^{P,n+1} = \mathbf{F}^{P,n}$, and $\mathbf{F}^{E,n+1} = \tilde{\mathbf{F}}^E$. However, if the yield condition is violated when $\boldsymbol{\tau}$ is computed from the trial deformation $\tilde{\mathbf{F}}^E$, then $\tilde{\mathbf{F}}^E$ must be modified accordingly to satisfy the constraint. This process is often referred to as the return mapping: $\tilde{\mathbf{F}}^E \to \mathbf{F}^{E,n+1}$. There are infinitely many ways that this can be done. We use associative plastic flow since it is straightforward with our choice of hyperelastic potential, and guarantees no violation of the second law of thermodynamics. See Section 3 for details.

Associativity requires that the projection of the stress be done in a direction equal to the elasticity tensor $\mathcal{C} = 2\mu\mathcal{I} + \lambda\mathbf{I}\otimes\mathbf{I}$ times the normal to the yield surface $\frac{\partial y}{\partial \boldsymbol{\tau}}$. Here $\mathcal{C}$ is a fourth-order tensor, $\mathcal{I}$ the fourth-order identity tensor, and $\mathbf{I}$ the second-order identity tensor. This process can be described succinctly in terms of the trial and project elastic Hencky strain as

$$\tilde{\boldsymbol{\epsilon}}^E - \boldsymbol{\epsilon}^{E,n+1} = \delta\frac{\partial y}{\partial \boldsymbol{\tau}}(\mathcal{C} : \boldsymbol{\epsilon}^{E,n+1}), \tag{1}$$

where $\tilde{\boldsymbol{\epsilon}}^E = \frac{1}{2}\ln(\tilde{\mathbf{F}}^E(\tilde{\mathbf{F}}^E)^T)$ is the trial elastic Hencky strain, $\boldsymbol{\epsilon}^{E,n+1} = \frac{1}{2}\ln(\mathbf{F}^{E,n+1}(\mathbf{F}^{E,n+1})^T)$ is the projected elastic Hencky strain, $\mathcal{C} : \boldsymbol{\epsilon}^{E,n+1} = \boldsymbol{\tau} = \lambda\mathrm{tr}(\boldsymbol{\epsilon}^{E,n+1})\mathbf{I} + 2\mu\boldsymbol{\epsilon}^{E,n+1}$ is the elasticity tensor, and $\delta > 0$ is a Lagrange multiplier chosen so that $\boldsymbol{\epsilon}^{E,n+1}$ is on the yield surface.

Due to our assumption of isotropy, the constraint in Equation (1) can be satisfied in terms of the singular values of the elastic deformation gradient. Furthermore, the singular vectors of the trial elastic strain do not change in the return mapping:

$$\mathbf{F}^{E,n+1} = \mathbf{U}^E\boldsymbol{\Sigma}^{E,n+1}(\mathbf{V}^E)^T,\ \tilde{\mathbf{F}}^E = \mathbf{U}^E\tilde{\boldsymbol{\Sigma}}^E(\mathbf{V}^E)^T. \tag{2}$$

With this convention, the trial and projected Hencky strains and Kirchhoff stresses satisfy

$$\tilde{\boldsymbol{\epsilon}}^E = \mathbf{U}^E\ln\tilde{\boldsymbol{\Sigma}}^E(\mathbf{U}^E)^T \tag{3}$$

$$\boldsymbol{\epsilon}^{E,n+1} = \mathbf{U}^E\ln\boldsymbol{\Sigma}^{E,n+1}(\mathbf{U}^E)^T \tag{4}$$

and

$$\tilde{\boldsymbol{\tau}}^E = \mathbf{U}^E\left(\lambda\mathrm{tr}(\ln(\tilde{\boldsymbol{\Sigma}}^E))\mathbf{I} + 2\mu\ln(\tilde{\boldsymbol{\Sigma}}^E)\right)(\mathbf{U}^E)^T \tag{5}$$

$$\boldsymbol{\tau}^{E,n+1} = \mathbf{U}^E\left(\lambda\mathrm{tr}(\ln(\boldsymbol{\Sigma}^{E,n+1}))\mathbf{I} + 2\mu\ln(\boldsymbol{\Sigma}^{E,n+1})\right)(\mathbf{U}^E)^T, \tag{6}$$

respectively.

The return mapping is completed as an operation on the eigenvalues $\tilde{\boldsymbol{\epsilon}}^E$. For simplicity of notation, we henceforth denote the eigenvalues of $\tilde{\boldsymbol{\epsilon}}^E$ and $\tilde{\boldsymbol{\tau}}^E$ by $\hat{\boldsymbol{\epsilon}}$ and $\hat{\boldsymbol{\tau}} = \lambda(\mathbf{1}\cdot\hat{\boldsymbol{\epsilon}})\mathbf{1} + 2\mu\hat{\boldsymbol{\epsilon}}$ respectively, where $\mathbf{1}$ is the vector of all ones. Furthermore, we refer to the eigenvalues of the projected $\boldsymbol{\epsilon}^{E,n+1}$ and $\boldsymbol{\tau}^{E,n+1}$ as $\mathrm{proj}(\hat{\boldsymbol{\epsilon}})$ and $\mathrm{proj}(\hat{\boldsymbol{\tau}})$ respectively. The process of satisfying Equation (1) is, in the case of the Rankine yield condition,

- If $\lambda \mathbf{1} \cdot \hat{\boldsymbol{\epsilon}} + 2\mu\epsilon_1 \leq \tau_C$, no projection, $\text{proj}(\hat{\boldsymbol{\epsilon}}) = \hat{\boldsymbol{\epsilon}}$,
- If $(2\mu + \lambda)\epsilon_2 + \lambda(\mathbf{1} \cdot \hat{\boldsymbol{\epsilon}} - \epsilon_1) \leq \tau_C < \lambda \mathbf{1} \cdot \hat{\boldsymbol{\epsilon}} + 2\mu\epsilon_1$, $\text{proj}(\hat{\boldsymbol{\epsilon}}) = \left(\frac{\tau_C - \lambda(\mathbf{1}\cdot\hat{\boldsymbol{\epsilon}} - \epsilon_1)}{2\mu+\lambda}, \epsilon_2, \epsilon_3\right)$,
- If $(2\mu+3\lambda)\epsilon_3 \leq \tau_C < (2\mu+\lambda)\epsilon_2 + \lambda(\mathbf{1}\cdot\hat{\boldsymbol{\epsilon}} - \epsilon_1)$, $\text{proj}(\hat{\boldsymbol{\epsilon}}) = \left(\frac{\tau_C - \lambda(\mathbf{1}\cdot\hat{\boldsymbol{\epsilon}} - \epsilon_1 - \epsilon_2)}{2\mu+2\lambda}, \frac{\tau_C - \lambda(\mathbf{1}\cdot\hat{\boldsymbol{\epsilon}} - \epsilon_1 - \epsilon_2)}{2\mu+2\lambda}, \epsilon_3\right)$,
- If $\tau_C < (2\mu + 3\lambda)\epsilon_3$, $\text{proj}(\hat{\boldsymbol{\epsilon}}) = \frac{\tau_C}{2\mu+3\lambda}\mathbf{1}$.

In the case of the von Mises yield condition, the projection is

- If $|\hat{\boldsymbol{\tau}} - \mathbf{1}\cdot\hat{\boldsymbol{\tau}}\mathbf{1}| \leq \tau_C$, no projection, $\text{proj}(\hat{\boldsymbol{\epsilon}}) = \hat{\boldsymbol{\epsilon}}$,
- If $|\hat{\boldsymbol{\tau}} - \mathbf{1}\cdot\hat{\boldsymbol{\tau}}\mathbf{1}| > \tau_C$, $\mathbf{p} = (\hat{\boldsymbol{\tau}} \cdot \mathbf{1})\frac{\mathbf{1}}{3}$, $\mathbf{d} = \hat{\boldsymbol{\tau}} - \mathbf{p}$, $\text{proj}(\hat{\boldsymbol{\tau}}) = \mathbf{p} + \tau_C \frac{\mathbf{d}}{|\mathbf{d}|}$, $\text{proj}(\hat{\boldsymbol{\epsilon}}) = \hat{\mathcal{C}}^{-1}\text{proj}(\hat{\boldsymbol{\tau}})$

where

$$\hat{\mathcal{C}} = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda \\ \lambda & 2\mu+\lambda & \lambda \\ \lambda & \lambda & 2\mu+\lambda \end{pmatrix}. \tag{7}$$

After the projection has been done, the singular values of the time t^{n+1} elastic deformation gradient are computed from $\boldsymbol{\Sigma}^{E,n+1} = \exp(\text{proj}(\hat{\boldsymbol{\epsilon}}))$, which are used to construct the deformation gradient as in Equation (2). Lastly, the time t^{n+1} plastic deformation gradient is computed from $\mathbf{F}^{P,n+1} = (\mathbf{F}^{E,n+1})^{-1}\mathbf{F}^{n+1}$.

2 Rates of plastic flow

The rate of change of the plastic decomposition $\mathbf{F} = \mathbf{F}_E\mathbf{F}_P$ is

$$\begin{aligned} \dot{\mathbf{F}} &= \dot{\mathbf{F}}_E\mathbf{F}_P + \mathbf{F}_E\dot{\mathbf{F}}_P \\ \dot{\mathbf{F}}_E &= \dot{\mathbf{F}}\mathbf{F}_P^{-1} - \mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}_P^{-1} \\ \dot{\mathbf{F}}_P &= {\mathbf{F}_E}^{-1}\dot{\mathbf{F}} - {\mathbf{F}_E}^{-1}\dot{\mathbf{F}}_E\mathbf{F}_P. \end{aligned} \tag{8}$$

Furthermore defining $\mathbf{b}_E = \mathbf{F}_E\mathbf{F}_E^T$ as the elastic right Cauchy-Green strain and using Equations (8), we can see that

$$\dot{\mathbf{b}}_E = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{b}_E + \mathbf{b}_E\frac{\partial \mathbf{v}}{\partial \mathbf{x}}^T + \mathcal{L}_{\mathbf{v}}\mathbf{b}_E \tag{9}$$

where

$$\mathcal{L}_{\mathbf{v}}\mathbf{b}_E = \mathbf{F}\dot{\mathbf{C}}_p^{-1}\mathbf{F}^T = -\mathbf{F}\mathbf{C}_p^{-1}\dot{\mathbf{C}}_p\mathbf{C}_p^{-1}\mathbf{F}^T = -\mathbf{F}\mathbf{C}_p^{-1}\dot{\mathbf{F}}_P^T\mathbf{F}_P\mathbf{C}_p^{-1}\mathbf{F}^T - \mathbf{F}\mathbf{C}_p^{-1}\mathbf{F}_P^T\dot{\mathbf{F}}_P\mathbf{C}_p^{-1}\mathbf{F}^T \tag{10}$$

and $\mathbf{C}_p = \mathbf{F}_P^T\mathbf{F}_P$ is the plastic left Cauchy-Green strain. It is convenient to use the notation

$$\dot{\mathbf{b}}_E = \dot{\mathbf{b}}_E|_{\dot{\mathbf{F}}_P=\mathbf{0}} + \mathcal{L}_{\mathbf{v}}\mathbf{b}_E,\ \dot{\mathbf{b}}_E|_{\dot{\mathbf{F}}_P=\mathbf{0}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{b}_E + \mathbf{b}_E\frac{\partial \mathbf{v}}{\partial \mathbf{x}}^T \tag{11}$$

The elastic Hencky strain $\boldsymbol{\epsilon}_E$ is defined as

$$\boldsymbol{\epsilon}_E = \frac{1}{2}\log(\mathbf{b}_E). \tag{12}$$

See Appendix (§5) for more details on functions defined this way. The rate of change of the elastic Hencky strain is given by

$$\dot{\boldsymbol{\epsilon}}_E = \left([\mathbf{B}](\mathbf{b}_E) \circ [\dot{\mathbf{b}}_E]\right)_{kl} \mathbf{u}_k \otimes \mathbf{u}_l \tag{13}$$

where $\mathbf{b}_E = \sum_i \lambda_i^{E^2} \mathbf{u}_i \otimes \mathbf{u}_i$, $\boldsymbol{\epsilon}_E = \sum_i \log\left(\lambda_i^E\right) \mathbf{u}_i \otimes \mathbf{u}_i$, $[\dot{\mathbf{b}}_E]_{ij} = \mathbf{u}_i \cdot \left(\dot{\mathbf{b}}_E \mathbf{u}_j\right)$ are the components of $\dot{\mathbf{b}}_E$ in the eigen basis and

$$[\mathbf{B}](\mathbf{b}_E) = \begin{pmatrix} \frac{1}{2\lambda_{E1}^2} & \frac{\log(\lambda_{E1})-\log(\lambda_{E2})}{\lambda_{E1}^2-\lambda_{E2}^2} & \frac{\log(\lambda_{E1})-\log(\lambda_{E3})}{\lambda_{E1}^2-\lambda_{E3}^2} \\ \frac{\log(\lambda_{E2})-\log(\lambda_{E1})}{\lambda_{E2}^2-\lambda_{E1}^2} & \frac{1}{2\lambda_{E2}^2} & \frac{\log(\lambda_{E2})-\log(\lambda_{E3})}{\lambda_{E2}^2-\lambda_{E3}^2} \\ \frac{\log(\lambda_{E3})-\log(\lambda_{E1})}{\lambda_{E3}^2-\lambda_{E1}^2} & \frac{\log(\lambda_{E3})-\log(\lambda_{E2})}{\lambda_{E3}^2-\lambda_{E2}^2} & \frac{1}{2\lambda_{E3}^2} \end{pmatrix}. \tag{14}$$

See Appendix (§5) and Equation (42) for the derivation.

2.1 Energy dissipation and stress/rate pairs

The energy at time t of the material in B^0 is

$$E(t; B^0) = \int_{B^0} \frac{R(\mathbf{X}, 0)}{2} |\mathbf{V}(\mathbf{X}, t)|_2^2 \, d\mathbf{X} + \int_{B^0} \psi(\mathbf{F}_E(\mathbf{X}, t), \mathbf{F}_P(\mathbf{X}, t)) d\mathbf{X}. \tag{15}$$

with $\mathbf{P} = \frac{\partial \psi}{\partial \mathbf{F}_E}(\mathbf{F}_E, \mathbf{F}_P)\mathbf{F}_P^{-T}$. The rate of change of the energy is

$$E'(t; B^0) = \int_{B^0} R(\mathbf{X}, 0)\mathbf{V}(\mathbf{X}, t)\mathbf{A}(\mathbf{X}, t) d\mathbf{X} + \int_{B^0} \frac{\partial \psi}{\partial \mathbf{F}_E}(\mathbf{F}_E(\mathbf{X}, t), \mathbf{F}_P(\mathbf{X}, t)) : \dot{\mathbf{F}}_E(\mathbf{X}, t) d\mathbf{X} + \tag{16}$$

$$\int_{B^0} \frac{\partial \psi}{\partial \mathbf{F}_P}(\mathbf{F}_E(\mathbf{X}, t), \mathbf{F}_P(\mathbf{X}, t)) : \dot{\mathbf{F}}_P(\mathbf{X}, t) d\mathbf{X}. \tag{17}$$

where

$$\int_{B^0} \frac{\partial \psi}{\partial \mathbf{F}_E}(\mathbf{F}_E, \mathbf{F}_P) : \dot{\mathbf{F}}_E d\mathbf{X} =$$

$$\int_{B^0} \frac{\partial \psi}{\partial \mathbf{F}_E}(\mathbf{F}_E, \mathbf{F}_P) : \left(\dot{\mathbf{F}}\mathbf{F}_P^{-1} - \mathbf{F}_E \dot{\mathbf{F}}_P \mathbf{F}_P^{-1}\right) d\mathbf{X} =$$

$$\int_{B^0} \mathbf{P} : \left(\dot{\mathbf{F}} - \mathbf{F}_E \dot{\mathbf{F}}_P\right) d\mathbf{X} =$$

$$-\int_{B^0} \mathbf{V} \cdot \left(\nabla^{\mathbf{X}} \cdot \mathbf{P}\right) d\mathbf{X} + \int_{\partial B^0} \mathbf{V} \cdot (\mathbf{P}\mathbf{N}) \, ds(\mathbf{X}) - \int_{B^0} \left(\mathbf{F}_E^T \mathbf{P}\right) : \dot{\mathbf{F}}_P d\mathbf{X} =$$

and using $R(\mathbf{X}, 0)\mathbf{A}(\mathbf{X}, t) = (\nabla^{\mathbf{X}} \cdot \mathbf{P})(\mathbf{X}, t)$ with Equation (16) gives

$$E'(t; B^0) = \int_{\partial B^0} \mathbf{V} \cdot (\mathbf{P}\mathbf{N}) \, ds(\mathbf{X}) - \int_{B^0} \left(\mathbf{F}_E^T \mathbf{P}\right) : \dot{\mathbf{F}}_P d\mathbf{X} + \int_{B^0} \frac{\partial \psi}{\partial \mathbf{F}_P}(\mathbf{F}_E, \mathbf{F}_P) : \dot{\mathbf{F}}_P d\mathbf{X}. \tag{18}$$

Note that $\mathbf{P} = \frac{\partial \psi}{\partial \mathbf{F}_E}(\mathbf{F}_E, \mathbf{F}_P)\mathbf{F}_P^{-T}$, $\left(\mathbf{F}_E^T \mathbf{P}\right) : \dot{\mathbf{F}}_P = \left(\mathbf{F}_E^T \frac{\partial \psi}{\partial \mathbf{F}_E}(\mathbf{F}_E, \mathbf{F}_P)\right) : \left(\dot{\mathbf{F}}_P \mathbf{F}_P^{-1}\right)$. The term

$$\mathbf{L}^P = \dot{\mathbf{F}}_P \mathbf{F}_P^{-1} \tag{19}$$

is called the plastic velocity gradient. Using this we can write the change in energy as

$$E'(t;B^0) = \int_{\partial B^0} \mathbf{V}\cdot(\mathbf{P}\mathbf{N})\,ds(\mathbf{X}) - \int_{B^0} \mathbf{M}^E : \mathbf{L}^P d\mathbf{X} + \int_{B^0} \frac{\partial \psi}{\partial \mathbf{F}_P}(\mathbf{F}_E,\mathbf{F}_P) : \dot{\mathbf{F}}_P d\mathbf{X}. \tag{20}$$

where we define the Mendel stress $\mathbf{M}^E$ as

$$\mathbf{M}^E = \mathbf{F}_E^T \frac{\partial \psi}{\partial \mathbf{F}_E}(\mathbf{F}_E,\mathbf{F}_P). \tag{21}$$

The term $\int_{\partial B^0} \mathbf{V}\cdot(\mathbf{P}\mathbf{N})\,ds(\mathbf{X})$ is the rate of work done on B^0 at time t via contact with material external to the region.

2.2 Isotropy

In this case we assume the energy density $\psi(\mathbf{F}_E,\mathbf{F}_P)$ is of the form $\psi(\mathbf{F}_E,\mathbf{F}_P) = \hat{\psi}(I(\mathbf{F}_E), II(\mathbf{F}_E), III(\mathbf{F}_E))$ and

$$\frac{\partial \psi}{\partial \mathbf{F}_E}(\mathbf{F}_E,\mathbf{F}_P) = \alpha\mathbf{F}_E + \beta\mathbf{b}_E\mathbf{F}_E + \gamma\mathbf{F}_E^{-T}$$

$$\boldsymbol{\tau} = \mathbf{P}\mathbf{F}^T = \frac{\partial \psi}{\partial \mathbf{F}_E}(\mathbf{F}_E,\mathbf{F}_P)\mathbf{F}_P^{-T}\mathbf{F}^T = \alpha\mathbf{b}_E + \beta{\mathbf{b}_E}^2 + \gamma\mathbf{I}.$$

Note that $\boldsymbol{\tau}$ and $\mathbf{b}_E$ as well as $\boldsymbol{\tau}$ and ${\mathbf{b}_E}^{-1}$ commute in this case

$$\mathbf{b}_E\boldsymbol{\tau} = \mathbf{b}_E\boldsymbol{\tau},\ \ {\mathbf{b}_E}^{-1}\boldsymbol{\tau} = \boldsymbol{\tau}{\mathbf{b}_E}^{-1}. \tag{22}$$

$\boldsymbol{\tau}$ is the Kirchhoff stress and it is convenient to work with for some models. E.g. we can rewrite the plastic dissipation in terms of $\boldsymbol{\tau}$ since

$$\mathbf{M}^E : \mathbf{L}^P = \boldsymbol{\tau} : \left(\mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}^{-1}\right). \tag{23}$$

Using the definitions in Equations (9) and (10) and

$$\mathcal{L}_{\mathbf{v}}\mathbf{b}_E\mathbf{b}_E^{-1} = -\mathbf{F}_E\mathbf{F}_P^{-T}\dot{\mathbf{F}}_P^T\mathbf{F}_E^{-1} - \mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}^{-1} \tag{24}$$

we can conclude that in the case of isotropic energy density

$$\begin{aligned}
\boldsymbol{\tau} : (\mathcal{L}_{\mathbf{v}}\mathbf{b}_E\mathbf{b}_E^{-1}) &= -\boldsymbol{\tau} : \left(\mathbf{F}_E\mathbf{F}_P^{-T}\dot{\mathbf{F}}_P^T\mathbf{F}_E^{-1}\right) - \boldsymbol{\tau} : \left(\mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}^{-1}\right) \\
&= -\mathrm{tr}\left(\boldsymbol{\tau}\mathbf{F}_E\mathbf{F}_P^{-T}\dot{\mathbf{F}}_P^T\mathbf{F}_E^{-1}\right) - \boldsymbol{\tau} : \left(\mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}^{-1}\right) \\
&= -\mathrm{tr}\left(\boldsymbol{\tau}\mathbf{F}_E\mathbf{F}_E^T\mathbf{F}_E^{-T}\mathbf{F}_P^{-T}\dot{\mathbf{F}}_P^T\mathbf{F}_E^{-1}\right) - \boldsymbol{\tau} : \left(\mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}^{-1}\right) \\
&= -\mathrm{tr}\left(\boldsymbol{\tau}\mathbf{b}_E\mathbf{F}^{-T}\dot{\mathbf{F}}_P^T\mathbf{F}_E^{-1}\right) - \boldsymbol{\tau} : \left(\mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}^{-1}\right) \\
&= -\mathrm{tr}\left(\mathbf{b}_E\boldsymbol{\tau}\mathbf{F}^{-T}\dot{\mathbf{F}}_P^T\mathbf{F}_E^{-1}\right) - \boldsymbol{\tau} : \left(\mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}^{-1}\right) \\
&= -\mathrm{tr}\left(\boldsymbol{\tau}\mathbf{F}^{-T}\dot{\mathbf{F}}_P^T\mathbf{F}_E^{-T}\right) - \boldsymbol{\tau} : \left(\mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}^{-1}\right) \\
&= -2\boldsymbol{\tau} : \left(\mathbf{F}_E\dot{\mathbf{F}}_P\mathbf{F}^{-1}\right)
\end{aligned} \tag{25}$$

2.3 Plastic dissipation rate without hardening

In summary, the rate of energy release due to plasticity (with no hardening) can be written as

$$\int_{B^0} \dot{w}_p(\mathbf{X}, t) d\mathbf{X} \tag{26}$$

where

$$\dot{w}_p = \left(\mathbf{F}_E^T \mathbf{P}\right) : \dot{\mathbf{F}}_P = \mathbf{M}^E : \mathbf{L}^p = -\frac{1}{2}\boldsymbol{\tau} : \left(\mathcal{L}_{\mathbf{v}} \mathbf{b}_E \mathbf{b}_E^{-1}\right) \tag{27}$$

where the last equality only holds for isotropic energy density.

3 Associative

Assume we have no hardening, e.g. $\tilde{\psi}(\mathbf{F}_E) = \hat{\psi}(\frac{1}{2}\left(\mathbf{F}_E{}^T\mathbf{F}_E - \mathbf{I}\right))$, thus $\mathbf{P} = \mathbf{F}_E \frac{\partial\hat{\psi}}{\partial \mathbf{E}^E}(\frac{1}{2}\left(\mathbf{F}_E{}^T\mathbf{F}_E - \mathbf{I}\right))\mathbf{F}_P^{-T}$ and the Mendel stress $\mathbf{M}^E$ satisfies

$$\mathbf{M}^E = \mathbf{F}_E^T \frac{\partial \psi}{\partial \mathbf{F}_E} = \mathbf{C}_E \frac{\partial \hat{\psi}}{\partial \mathbf{E}^E}. \tag{28}$$

If we choose $\mathbf{L}^P$ such that

$$\mathbf{M}^E : \mathbf{L}^P \geq \mathbf{M}^* : \mathbf{L}^P \tag{29}$$

for all admissible states of stress $\mathbf{M}^*$, then

1. If $\mathbf{M}^* = \mathbf{0}$ is an admissible state of stress, then

$$E'(t; B^0) \leq \int_{\partial B^0} \mathbf{V} \cdot (\mathbf{P}\mathbf{N})\, ds(\mathbf{X}) \tag{30}$$

   which says that the plasticity dissipates energy.

2. If the region of admissible $\mathbf{M}^*$ is (a) convex and (b) defined via $f(\mathbf{M}^*) \leq 0$ then $\mathbf{L}^P \in \partial f(\mathbf{M}^E)$ satisfies Equation (29).

Similarly, if we choose $-\frac{1}{2}\mathcal{L}_{\mathbf{v}}\mathbf{b}_E\mathbf{b}_E^{-1} \in \partial f$ we get an associative plastic flow when we write the yield surface in terms of $\boldsymbol{\tau}$: $f(\boldsymbol{\tau})$.

3.1 Yield surface and plastic flow

We will have plastic flow $\dot{\mathbf{F}}_P \neq \mathbf{0}$ when our stress is on the boundary of the feasible region, and without plasticity we would leave the region. In the case of isoptropy and a yield surface defined in terms of the Kirchhoff stress, then

$$\mathcal{L}_{\mathbf{v}} \mathbf{b}_E = -2\lambda \frac{\partial f}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau})\mathbf{b}_E \tag{31}$$

where

- If $f(\boldsymbol{\tau}) < 0$ or $f(\boldsymbol{\tau}) = 0$ and $\alpha \leq 0$, then $\lambda = 0$.
- Otherwise if, $f(\boldsymbol{\tau}) = 0$ and $\alpha > 0$, then $\lambda = \frac{\alpha}{\beta}$

where

$$\alpha = \frac{\partial f}{\partial \boldsymbol{\tau}} : \frac{\partial \boldsymbol{\tau}}{\partial \mathbf{b}_E} : \dot{\mathbf{b}}_E|_{\dot{\mathbf{F}}_P = \mathbf{0}}, \; \beta = 2\frac{\partial f}{\partial \boldsymbol{\tau}} : \frac{\partial \boldsymbol{\tau}}{\partial \mathbf{b}_E} : \left(\frac{\partial f}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau})\mathbf{b}_E\right). \tag{32}$$

3.2 Isoptropic yield surface

Suppose the yield surface function $f : \mathcal{V}^2_{\text{sym}} \to \mathbb{R}$ is isotropic: $f(\mathbf{V}\boldsymbol{\tau}\mathbf{V}^T) = f(\boldsymbol{\tau})$ for all rotations $\mathbf{V}$. Then as discussed in Appendix (§5.2), we can write $f(\boldsymbol{\tau}) = \hat{f}(\tau_1, \tau_2, \tau_3)$ where $\boldsymbol{\tau} = \sum_i \tau_i \mathbf{u}_i \otimes \mathbf{u}_i$ and $\frac{\partial f}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau}) = \sum_i \frac{\partial \hat{f}}{\partial \tau_i} \mathbf{u}_i \otimes \mathbf{u}_i$. Therefore since $\boldsymbol{\tau}$ and $\mathbf{b}_E$ have the same eigenvectors

$$\frac{\partial f}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau})\mathbf{b}_E = \sum_i \frac{\partial \hat{f}}{\partial \tau_i} \lambda^2_{Ei} \mathbf{u}_i \otimes \mathbf{u}_i \tag{33}$$

Furthermore using the properties of isotropic energy density,

$$\beta = 2 \sum_{i,j} \frac{\partial \hat{f}}{\partial \tau_i} \tilde{C}_{ij}(\mathbf{b}_E) \frac{\partial \hat{f}}{\partial \tau_j} \lambda^2_{Ej} \tag{34}$$

where

$$\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{b}_E}(\mathbf{b}_E) : \left(\sum_j \sigma_j \mathbf{u}_j \otimes \mathbf{u}_j \right) = \sum_{i,j} \tilde{C}_{ij}(\mathbf{b}_E) \sigma_j \mathbf{u}_i \otimes \mathbf{u}_i$$

for arbitrary $\sum_j \sigma_j \mathbf{u}_j \otimes \mathbf{u}_j$.

4 Hencky Strain

If we define the elastic potential as a function of the Hencky strain as

$$\psi(\mathbf{F}_E, \mathbf{F}_P) = \mu \boldsymbol{\epsilon}_E : \boldsymbol{\epsilon}_E + \frac{\lambda}{2} \text{tr}\left(\boldsymbol{\epsilon}_E\right)^2 \tag{35}$$

then

$$\boldsymbol{\tau} = \boldsymbol{C}\boldsymbol{\epsilon}_E = 2\mu \boldsymbol{\epsilon}_E + \lambda \text{tr}\left(\boldsymbol{\epsilon}_E\right) \mathbf{I}. \tag{36}$$

This can be written in terms of the eigen basis of $\mathbf{b}_E$ as

$$\boldsymbol{\tau} = \boldsymbol{C}\boldsymbol{\epsilon}_E = \sum_{i,j} \hat{C}_{ij} \log\left(\lambda^E_j\right) \mathbf{u}_i \otimes \mathbf{u}_i \tag{37}$$

with

$$[\hat{\mathbf{C}}] = \begin{pmatrix} 2\mu + \lambda & \lambda & \lambda \\ \lambda & 2\mu + \lambda & \lambda \\ \lambda & \lambda & 2\mu + \lambda \end{pmatrix}.$$

4.1 Yield surface and plastic rate of change

With this energy density, the rate of change of the elastic Hencky strain has the favorable property that its direction is simply related to the yield surface when it is written in terms of $\boldsymbol{\epsilon}_E$. Specifically, α and β in Equation (32) can be written as

$$\alpha = \frac{\partial f}{\partial \boldsymbol{\tau}} : \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\epsilon}_E} : \dot{\boldsymbol{\epsilon}}_E|_{\dot{\mathbf{F}}_P = \mathbf{0}}, \ \beta = 2 \frac{\partial f}{\partial \boldsymbol{\tau}} : \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\epsilon}_E} : \left(\left([\mathbf{B}](\mathbf{b}_E) \circ [\frac{\partial f}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau})\mathbf{b}_E] \right)_{kl} \mathbf{u}_k \otimes \mathbf{u}_l \right) \tag{38}$$

and since $\frac{\partial f}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau}) = \sum_i \frac{\partial \hat{f}}{\partial \tau_i} \mathbf{u}_i \otimes \mathbf{u}_i$ and $\mathbf{b}_E = \sum_i \lambda_{Ei}^2 \mathbf{u}_i \otimes \mathbf{u}_i$ and

$$[\frac{\partial f}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau})\mathbf{b}_E]_{ij} = \mathbf{u}_i \cdot \left(\frac{\partial f}{\partial \boldsymbol{\tau}}(\boldsymbol{\tau})\mathbf{b}_E \mathbf{u}_j\right) = \begin{cases} \frac{\partial \hat{f}}{\partial \tau_i}\lambda_{Ei}^2, & i = j \\ 0, & \text{otherwise} \end{cases}$$

and $[\mathbf{B}](\mathbf{b}_E)$ from Equation (14), as well as $\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\epsilon}_E} = \boldsymbol{C}$ from Equation (37)

$$\beta = 2\sum_{i,j} \frac{\partial \hat{f}}{\partial \tau_i} \hat{C}_{ij} \frac{\partial \hat{f}}{\partial \tau_j} \tag{39}$$

5 Appendix: eigen decomposition differentials

Consider the space of symmetric 3×3 matrices $\mathbb{R}^{3\times 3}_{\text{sym}}$, thus $\mathbf{S} \in \mathbb{R}^{3\times 3}_{\text{sym}}$ have eigen decompositions, $\mathbf{S} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T$ for some orthogonal $\mathbf{V}$ and diagonal $\boldsymbol{\Lambda}$. We can define a class of functions $\mathbf{g} : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}^{3\times 3}_{\text{sym}}$ that are inherited from scalar functions $g : \mathbb{R} \to \mathbb{R}$ as

$$\mathbf{g}(\mathbf{S}) = \mathbf{V}g(\boldsymbol{\Lambda})\mathbf{V}^T$$

where we use the notation

$$g(\boldsymbol{\Lambda}) = \begin{pmatrix} g(\lambda_1) & & \\ & g(\lambda_2) & \\ & & g(\lambda_3) \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}.$$

We can derive the differentials of scalar inherited $\mathbf{g}$ using the expressions for the differentials of the eigen decomposition of $\mathbf{S}$. The eigen decomposition of the symmetric matrix $\mathbf{S}$ can be thought of as a function over $\mathbb{R}^{3\times 3}_{\text{sym}}$: $\mathbf{V} : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}^{3\times 3}_{\text{orth}}$ and $\boldsymbol{\Lambda} : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}^{3\times 3}_{\text{diag}}$, or $\mathbf{V}(\mathbf{S})$ and $\boldsymbol{\Lambda}(\mathbf{S})$ to emphasize the dependent variable. By definition, we have the relation

$$\delta\mathbf{S} = \delta\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T + \mathbf{V}\delta\boldsymbol{\Lambda}\mathbf{V}^T + \mathbf{V}\boldsymbol{\Lambda}\delta\mathbf{V}^T$$

and since $\mathbf{V}^T\mathbf{V} = \mathbf{I}$,

$$\delta\mathbf{V}^T\mathbf{V} + \mathbf{V}^T\delta\mathbf{V} = \mathbf{0}.$$

Using $\mathbf{W} = \delta\mathbf{V}^T\mathbf{V}$, we see that $\mathbf{W}$ is skew symmetric and that

$$\mathbf{V}^T\delta\mathbf{S}\mathbf{V} = \mathbf{W}^T\boldsymbol{\Lambda} + \delta\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\mathbf{W}.$$

Since $\mathbf{W}$ is skew symmetric, it can be written as

$$\mathbf{W} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

and thus

$$\mathbf{V}^T\delta\mathbf{S}\mathbf{V} = \begin{pmatrix} \delta\lambda_1 & -\omega_3(\lambda_2 - \lambda_1) & \omega_2(\lambda_3 - \lambda_1) \\ -\omega_3(\lambda_2 - \lambda_1) & \delta\lambda_2 & -\omega_1(\lambda_3 - \lambda_1) \\ \omega_2(\lambda_3 - \lambda_1) & -\omega_1(\lambda_3 - \lambda_2) & \delta\lambda_3 \end{pmatrix}. \tag{40}$$

Thus denoting $\mathbf{A} = \mathbf{V}^T \delta \mathbf{S} \mathbf{V}$, we have the expressions

$$\omega_1 = -\frac{a_{32}}{\lambda_3 - \lambda_2},\ \omega_2 = \frac{a_{31}}{\lambda_3 - \lambda_1},\ \omega_3 = -\frac{a_{21}}{\lambda_2 - \lambda_1},\ \text{and } \delta\lambda_i = a_{ii},\ i = 1, 2, 3$$

Similar to the eigen decomposition

$$\mathbf{V}^T \delta \mathbf{g} \mathbf{V} = \mathbf{W}^T g(\mathbf{\Lambda}) + \delta g(\mathbf{\Lambda}) + g(\mathbf{\Lambda}) \mathbf{W}$$

where

$$\delta g(\mathbf{\Lambda}) = \begin{pmatrix} g'(\lambda_1)\delta\lambda_1 & & \\ & g'(\lambda_2)\delta\lambda_2 & \\ & & g'(\lambda_3)\delta\lambda_3 \end{pmatrix}.$$

Thus,

$$\mathbf{V}^T \delta \mathbf{g} \mathbf{V} = \begin{pmatrix} g'(\lambda_1) a_{11} & \frac{g(\lambda_2)-g(\lambda_1)}{\lambda_2-\lambda_1} a_{21} & \frac{g(\lambda_3)-g(\lambda_1)}{\lambda_3-\lambda_1} a_{31} \\ \frac{g(\lambda_2)-g(\lambda_1)}{\lambda_2-\lambda_1} a_{21} & g'(\lambda_2) a_{22} & \frac{g(\lambda_3)-g(\lambda_2)}{\lambda_3-\lambda_2} a_{32} \\ \frac{g(\lambda_3)-g(\lambda_1)}{\lambda_3-\lambda_1} a_{31} & \frac{g(\lambda_3)-g(\lambda_2)}{\lambda_3-\lambda_2} a_{32} & g'(\lambda_3) a_{33} \end{pmatrix}$$

and

$$\delta \mathbf{g} = \mathbf{V} \begin{pmatrix} g'(\lambda_1) a_{11} & \frac{g(\lambda_2)-g(\lambda_1)}{\lambda_2-\lambda_1} a_{21} & \frac{g(\lambda_3)-g(\lambda_1)}{\lambda_3-\lambda_1} a_{31} \\ \frac{g(\lambda_2)-g(\lambda_1)}{\lambda_2-\lambda_1} a_{21} & g'(\lambda_2) a_{22} & \frac{g(\lambda_3)-g(\lambda_2)}{\lambda_3-\lambda_2} a_{32} \\ \frac{g(\lambda_3)-g(\lambda_1)}{\lambda_3-\lambda_1} a_{31} & \frac{g(\lambda_3)-g(\lambda_2)}{\lambda_3-\lambda_2} a_{32} & g'(\lambda_3) a_{33} \end{pmatrix} \mathbf{V}^T.$$

We can rewrite this in terms of the matrix

$$\mathbf{B} = \begin{pmatrix} g'(\lambda_1) & \frac{g(\lambda_2)-g(\lambda_1)}{\lambda_2-\lambda_1} & \frac{g(\lambda_3)-g(\lambda_1)}{\lambda_3-\lambda_1} \\ \frac{g(\lambda_2)-g(\lambda_1)}{\lambda_2-\lambda_1} & g'(\lambda_2) & \frac{g(\lambda_3)-g(\lambda_2)}{\lambda_3-\lambda_2} \\ \frac{g(\lambda_3)-g(\lambda_1)}{\lambda_3-\lambda_1} & \frac{g(\lambda_3)-g(\lambda_2)}{\lambda_3-\lambda_2} & g'(\lambda_3) \end{pmatrix}$$

using the Hadamard product (or entry-wise product) where the i, j entry of $\mathbf{A} \circ \mathbf{B}$ is $A_{ij} B_{ij}$ (with no summation on the repeated indices). That is,

$$\delta \mathbf{g} = \mathbf{V} \left(\mathbf{B} \circ \left(\mathbf{V}^T \delta \mathbf{S} \mathbf{V} \right) \right) \mathbf{V}^T \tag{41}$$

5.1 Symmetric tensors

This result generalizes to functions over symmetric tensors. If $\mathbf{g} : \mathcal{V}^2_{\text{sym}} \to \mathcal{V}^2_{\text{sym}}$, then

$$\delta \mathbf{g} = \left([\mathbf{B}](\mathbf{S}) \circ [\delta \mathbf{S}] \right)_{kl} \mathbf{u}_k \otimes \mathbf{u}_l \tag{42}$$

where $\mathbf{S} = \sum_i \lambda_i \mathbf{u}_i \times \mathbf{u}_i$ is the eigenvalue decomposition of $\mathbf{S}$. $[\delta\mathbf{S}], [\mathbf{B}](\mathbf{S}) \in \mathbb{R}^{3\times 3}$ and $[\mathbf{B}](\mathbf{S}) \circ [\delta \mathbf{S}] \in \mathbb{R}^{3\times 3}$ is their Hadamard product. The entries in the matrix $[\delta\mathbf{S}]$ are $[\delta\mathbf{S}]_{ij} = \mathbf{u}_i \cdot (\delta \mathbf{S} \mathbf{u}_j)$, i.e. it is the expression of $\delta\mathbf{S}$ in the eigenbasis of $\mathbf{S}$. We would assume the convention $\lambda_1 \geq \lambda_2 \geq \lambda_3$ to make the mapping $[\mathbf{B}] :\to \mathcal{V}^2_{\text{sym}}$ well defined from

$$[\mathbf{B}](\mathbf{S}) = \begin{pmatrix} g'(\lambda_1) & \frac{g(\lambda_2)-g(\lambda_1)}{\lambda_2-\lambda_1} & \frac{g(\lambda_3)-g(\lambda_1)}{\lambda_3-\lambda_1} \\ \frac{g(\lambda_2)-g(\lambda_1)}{\lambda_2-\lambda_1} & g'(\lambda_2) & \frac{g(\lambda_3)-g(\lambda_2)}{\lambda_3-\lambda_2} \\ \frac{g(\lambda_3)-g(\lambda_1)}{\lambda_3-\lambda_1} & \frac{g(\lambda_3)-g(\lambda_2)}{\lambda_3-\lambda_2} & g'(\lambda_3) \end{pmatrix}.$$

5.2 Scalar functions of symmetric tensors

Let $f : \mathcal{V}^2_{\text{sym}} \to \mathbb{R}$ with $f(\mathbf{S}) = \hat{f}(\lambda_1, \lambda_2, \lambda_3) = \tilde{f}(I(\mathbf{S}), II(\mathbf{S}), III(\mathbf{S}))$ where

$$I(\mathbf{S}) = \lambda_1 + \lambda_2 + \lambda_3,\ II(\mathbf{S}) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,\ III(\mathbf{S}) = \lambda_1\lambda_2\lambda_3. \tag{43}$$

Using Equation (40), we can conclude

$$\delta f = \frac{\partial f}{\partial \mathbf{S}}(\mathbf{S}) = \sum_i \frac{\partial \hat{f}}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_3)\delta\lambda_i = \sum_i \frac{\partial \hat{f}}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_3)\mathbf{u}_i \cdot (\delta\mathbf{S}\mathbf{u}_i)$$

Thus, the derivative is given by

$$\frac{\partial f}{\partial \mathbf{S}}(\mathbf{S}) = \sum_i \frac{\partial \hat{f}}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_3)\mathbf{u}_i \otimes \mathbf{u}_i. \tag{44}$$